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Jordan–Wigner fermionization of the one-dimensional Bariev model of three coupled XY chains

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Abstract

The Jordan–Wigner fermionization for the one-dimensional Bariev model of three coupled XY chains is formulated. The *L*-matrix in terms of fermion operators and the *R*-matrix are presented explicitly. Furthermore, the graded reflection equations and their solutions are discussed.

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It is well known that the Jordan–Wigner transformation is generally used to convert spin models into fermion models, or *vice versa*, in condensed matter physics. Successfully applied by Olmedilla *et al* [1] to the one-dimensional (1D) Hubbard model, the Jordan–Wigner transformation led to a better understanding of this model in the framework of the quantum inverse scattering method (QISM) [2, 3]. However, its generalization to an arbitrary number of internal degrees of freedom is not easy due to the cumbersome calculations involved. In this context, a general scheme is also proposed by Göhmann and Murakami [4] to treat the fermionization of integrable lattice systems, which was directly applied to some fundamental models, such as the XYZ spin chain, etc (see also [5]). Recently, another generalization of the Jordan–Wigner transformation was proposed by Batista and Ortiz [6] to convert spin-*S* operators of the $SU(2)$ algebra into fermion operators. Although it reveals a new insight into the integrable models of strongly correlated electrons, it can also be applied to higher spin representation. In spite of all this progress in the fermionization of spin chains, some important systems, such as the three coupled XY chains [7, 8] still lack comprehensive investigation by means of the graded QISM. In this paper, we generalize the Jordan–Wigner transformation [1, 9, 10] to the 1D Bariev model of three coupled XY chains [7, 8, 11], which possesses three internal degrees of freedom. The model has finite magnetization on the ground state in the zero external fields and exhibits the existence of hole pairs of the Cooper type which are relevant to theories of superconductors [11]. We convert the three coupled XY chains into a fermion model of strongly correlated electrons. The graded Yang–Baxter relation and the graded reflection equations, which guarantee the integrability of the model in the bulk and at the boundaries, respectively, are formulated in the framework of the QISM. The first conserved current next to the Hamiltonian and the boundary conditions in terms of fermion operators are constructed. Our results facilitate the algebraic Bethe ansatz [12] for the fermion model with periodic and open boundary conditions, which provides the spectrum of all conserved charges, essential for studying finite temperature properties of the integrable models.

We begin by considering a spin chain model defined by the following Hamiltonian

$$H = \sum_{j=1}^{L} H_{j,j+1} \tag{1}$$

where $H_{j,j+1}$ denotes the Hamiltonian density of three XY spin chains coupled to each other [8] as

$$H_{j,j+1} = \sum_{\alpha=1}^{3} \left(\sigma^+_{j(\alpha)}\sigma^-_{j+1(\alpha)} + \sigma^-_{j(\alpha)}\sigma^+_{j+1(\alpha)}\right) \exp\left(\eta \sum_{\alpha'\neq\alpha} \sigma^+_{j+\theta(\alpha'-\alpha)(\alpha')}\sigma^-_{j+\theta(\alpha'-\alpha)(\alpha')}\right) \tag{2}$$

where $\sigma^{\pm}_{j(\alpha)} = \frac{1}{2}\left(\sigma^x_{j(\alpha)} \pm \mathrm{i}\sigma^y_{j(\alpha)}\right)$, with $\sigma^x_{j(\alpha)}, \sigma^y_{j(\alpha)}$ and $\sigma^z_{j(\alpha)}$ being the usual Pauli spin operators at site j corresponding to the αth ($\alpha = 1, 2, 3$) XY spin chain, $\theta(\alpha' - \alpha)$ is a step function of $(\alpha' - \alpha)$ and η is a coupling constant. As is shown in [8], the Hamiltonian commutes with a one-parameter family of transfer matrix $\tau(u)$ of a two-dimensional lattice statistical mechanics model. This transfer matrix is the trace of a monodromy matrix $T(u)$, which is defined, as usual, by

$$T(u) = L_{0N}(u)\cdots L_{01}(u) \tag{3}$$

with $L_{0j}(u)$ of the form

$$L_{0j}(u) = L^{(1)}_{0j}(u)L^{(2)}_{0j}(u)L^{(3)}_{0j}(u) \tag{4}$$

where

$$\begin{aligned} L^{(\alpha)}_{0j}(u) = {} & \frac{1}{2}\left(1 + \sigma^z_{j(\alpha)}\sigma^z_{0(\alpha)}\right) + \frac{1}{2}u\left(1 - \sigma^z_{j(\alpha)}\sigma^z_{0(\alpha)}\right)\exp\left(\eta \sum_{\substack{\alpha'=1\\ \alpha'\neq\alpha}}^{3} \sigma^+_{0(\alpha')}\sigma^-_{0(\alpha')}\right) \\ & + \left(\sigma^-_{j(\alpha)}\sigma^+_{0(\alpha)} + \sigma^+_{j(\alpha)}\sigma^-_{0(\alpha)}\right)\sqrt{1 + \exp\left(2\eta \sum_{\substack{\alpha'=1\\ \alpha'\neq\alpha}}^{3} \sigma^+_{0(\alpha')}\sigma^-_{0(\alpha')}\right)u^2}. \end{aligned} \tag{5}$$

The explicit form of the corresponding R-matrix is given in [8].

Now let us introduce the following Jordan–Wigner transformation for a model with three degrees of freedom

$$\begin{pmatrix}\sigma^+_{j(\alpha)}\\ \sigma^-_{j(\alpha)}\end{pmatrix} = \left[V_{j(\alpha)}\right]^2 \begin{pmatrix}c^\dagger_{j(\alpha)}\\ c_{j(\alpha)}\end{pmatrix} \qquad \sigma^z_{j(\alpha)} = 2n_{j(\alpha)} - 1 \tag{6}$$

where

$$V_{j(\alpha)} = \begin{pmatrix} v_{j(\alpha)} & 0\\ 0 & v^{-1}_{j(\alpha)}\end{pmatrix} \tag{7}$$

with

$$v_{j(1)} = \exp\left(\frac{1}{2}\mathrm{i}\pi \sum_{i=1}^{j-1} c^\dagger_{i(1)}c_{i(1)}\right) \tag{8}$$

$$v_{j(2)} = \exp\left(\frac{1}{2}\mathrm{i}\pi\sum_{i=1}^{L} c^{\dagger}_{i(1)}c_{i(1)}\right)\exp\left(\frac{1}{2}\mathrm{i}\pi\sum_{i=1}^{j-1} c^{\dagger}_{i(2)}c_{i(2)}\right) \tag{9}$$

$$v_{j(3)} = \exp\left(\frac{1}{2}\mathrm{i}\pi\sum_{i=1}^{L} c^{\dagger}_{i(1)}c_{i(1)}\right)\exp\left(\frac{1}{2}\mathrm{i}\pi\sum_{i=1}^{L} c^{\dagger}_{i(2)}c_{i(2)}\right)\exp\left(\frac{1}{2}\mathrm{i}\pi\sum_{i=1}^{j-1} c^{\dagger}_{i(3)}c_{i(3)}\right). \tag{10}$$

Here $c^{\dagger}_{j(\alpha)}$ and $c_{j(\alpha)}$ are creation and annihilation operators with colour index α ($\alpha = 1, 2, 3$) satisfying the anti-commutation relations and $n_{j(\alpha)} = c^{\dagger}_{j(\alpha)}c_{j(\alpha)}$ is the density operator. Under such transformation, one may obtain the Hamiltonian of a fermionic model which is equivalent to the model (1)

$$H_{j,j+1} = \sum_{\alpha=1}^{3}\left(c^{\dagger}_{j(\alpha)}c_{j+1(\alpha)} + c^{\dagger}_{j+1(\alpha)}c_{j(\alpha)}\right)\exp\left[\eta\sum_{\alpha'\neq\alpha} n_{j+\theta(\alpha'-\alpha)(\alpha')}\right] \tag{11}$$

describing fermions hopping along a lattice with strong correlations determined by the occupation numbers of the fermions with different colours. In order to apply the QISM approach, let us now connect the fermion model (11) with an L-matrix which realizes the graded Yang–Baxter relation. For this purpose, let us first define the matrix

$$V_j = V_{j(1)} \otimes V_{j(2)} \otimes V_{j(3)}. \tag{12}$$

Then the fermionic L-matrix $\mathcal{L}(u)$ can be presented as

$$\mathcal{L}_{0j}(u) = V_{j+1}L_{0j}(u)V_j^{-1}. \tag{13}$$

After a lengthy algebra, we may write the fermionic L-matrix in the following way

$$\mathcal{L}_{0j}(u) = \mathcal{L}^{(1)}_{0j}(u)\mathcal{L}^{(2)}_{0j}(u)\mathcal{L}^{(3)}_{0j}(u) \tag{14}$$

where

$$\mathcal{L}^{(1)}_{0j}(u) = \begin{pmatrix}
g^{+}_{j(1)} & 0 & 0 & 0 & -f_1^2 c_{j(1)} & 0 & 0 & 0 \\
0 & \tilde{g}^{+}_{j(1)} & 0 & 0 & 0 & f_1 f_2 c_{j(1)} & 0 & 0 \\
0 & 0 & \tilde{g}^{+}_{j(1)} & 0 & 0 & 0 & f_1 f_2 c_{j(1)} & 0 \\
0 & 0 & 0 & \tilde{\tilde{g}}^{+}_{j(1)} & 0 & 0 & 0 & -f_2^2 c_{j(1)} \\
\mathrm{i} f_1^2 c^{\dagger}_{j(1)} & 0 & 0 & 0 & g^{-}_{j(1)} & 0 & 0 & 0 \\
0 & -\mathrm{i} f_1 f_2 c^{\dagger}_{j(1)} & 0 & 0 & 0 & \tilde{g}^{-}_{j(1)} & 0 & 0 \\
0 & 0 & -\mathrm{i} f_1 f_2 c^{\dagger}_{j(1)} & 0 & 0 & 0 & \tilde{g}^{-}_{j(1)} & 0 \\
0 & 0 & 0 & \mathrm{i} f_2^2 c^{\dagger}_{j(1)} & 0 & 0 & 0 & \tilde{\tilde{g}}^{-}_{j(1)}
\end{pmatrix}$$

$$\mathcal{L}^{(2)}_{0j}(u) = \begin{pmatrix}
g^{+}_{j(2)} & 0 & -\mathrm{i} f_1^2 c_{j(2)} & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{g}^{+}_{j(2)} & 0 & \mathrm{i} f_1 f_2 c_{j(2)} & 0 & 0 & 0 & 0 \\
f_1^2 c^{\dagger}_{j(2)} & 0 & g^{-}_{j(2)} & 0 & 0 & 0 & 0 & 0 \\
0 & -f_1 f_2 c^{\dagger}_{j(2)} & 0 & \tilde{g}^{-}_{j(2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{g}^{+}_{j(2)} & 0 & -\mathrm{i} f_1 f_2 c_{j(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{\tilde{g}}^{+}_{j(2)} & 0 & \mathrm{i} f_2^2 c_{j(2)} \\
0 & 0 & 0 & 0 & f_1 f_2 c^{\dagger}_{j(2)} & 0 & \tilde{g}^{-}_{j(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & -f_2^2 c^{\dagger}_{j(2)} & 0 & \tilde{\tilde{g}}^{-}_{j(2)}
\end{pmatrix}$$

$$\mathcal{L}^{(3)}_{0j}(u) = \begin{pmatrix}
g^{+}_{j(3)} & f_1^2 c_{j(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mathrm{i} f_1^2 c^{\dagger}_{j(3)} & g^{-}_{j(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{g}^{+}_{j(3)} & f_1 f_2 c_{j(3)} & 0 & 0 & 0 & 0 \\
0 & 0 & -\mathrm{i} f_1 f_2 c^{\dagger}_{j(3)} & \tilde{g}^{-}_{j(3)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{g}^{+}_{j(3)} & f_1 f_2 c_{j(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & -\mathrm{i} f_1 f_2 c^{\dagger}_{j(3)} & \tilde{g}^{-}_{j(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\tilde{g}}^{+}_{j(3)} & f_2^2 c_{j(3)} \\
0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i} f_2^2 c^{\dagger}_{j(3)} & \tilde{\tilde{g}}^{-}_{j(3)}
\end{pmatrix}.$$

Above we have introduced the notation

$$g^{+}_{j(\alpha)} = u\exp(2\eta) + (\mathrm{i} - u\exp(2\eta))n_{j(\alpha)} \qquad \tilde{g}^{+}_{j(\alpha)} = u\exp(\eta) + (\mathrm{i} - u\exp(\eta))n_{j(\alpha)}$$

$$\tilde{\tilde{g}}^{+}_{j(\alpha)} = u + (\mathrm{i} - u)n_{j(\alpha)} \qquad g^{-}_{j(\alpha)} = 1 - (1 + \mathrm{i}u\exp(2\eta))n_{j(\alpha)}$$

$$\tilde{g}^{-}_{j(\alpha)} = 1 - (1 + \mathrm{i}u\exp(\eta))n_{j(\alpha)} \qquad \tilde{\tilde{g}}^{-}_{j(\alpha)} = 1 - (1 + \mathrm{i}u)n_{j(\alpha)}$$

$$f_1 = \sqrt{1 + u^2\exp(2\eta)} \qquad f_2 = \sqrt{1 + u^2}.$$

After sophisticated algebra, indeed, we can incorporate the fermionic L-matrix (14) into the graded Yang–Baxter relation

$$\mathcal{R}(u, v)\mathcal{L}_{0j}(u) \otimes_s \mathcal{L}_{0j}(v) = \mathcal{L}_{0j}(v) \otimes_s \mathcal{L}_{0j}(u)\mathcal{R}(u, v) \tag{15}$$

which is essential to the integrability of the model (11). The fermion version of the R-matrix comprises

$$\mathcal{R}(u, v) = W \cdot R(u, v) \cdot W^{-1} \tag{16}$$

where W is a 64×64 diagonal matrix given by

$$W = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \otimes M \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

with $M = \mathrm{diag}\{1, -\mathrm{i}, -1, \mathrm{i}, -\mathrm{i}, 1, -\mathrm{i}, 1, -1, \mathrm{i}, -1, \mathrm{i}, \mathrm{i}, -1, -\mathrm{i}, 1\}$. $R(u, v)$ stands for the R-matrix for the spin model (2), which is presented in [8]. In (15) above, $\otimes_s$ denotes the graded tensor product

$$[A \otimes_s B]_{\alpha\beta,\gamma\delta} = (-1)^{[P(\alpha)+P(\gamma)]P(\beta)} A_{\alpha\gamma} B_{\beta\delta} \tag{18}$$

with the Grassmann parities obeying the grading $P(1) = P(4) = P(6) = P(7) = 0$ and $P(2) = P(3) = P(5) = P(8) = 1$. We would like to stress that the grading and the W-matrix are uniquely defined by the Jordan–Wigner transformation, only if the Jordan–Wigner transformation is used in the specific way of equation (6). However, one can see that the grading coincides with the choice of the bosonic and fermionic degrees of freedom as the up-spin is referred to as a bosonic degree of freedom, whereas down-spin is the fermionic degree of freedom on the basis of the auxiliary space $V(\cong C^2 \otimes C^2 \otimes C^2)$ as

$$e_1 = |\uparrow\uparrow\uparrow\rangle \quad e_2 = |\uparrow\uparrow\downarrow\rangle \quad e_3 = |\uparrow\downarrow\uparrow\rangle \quad e_4 = |\uparrow\downarrow\downarrow\rangle$$
$$e_5 = |\downarrow\uparrow\uparrow\rangle \quad e_6 = |\downarrow\uparrow\downarrow\rangle \quad e_7 = |\downarrow\downarrow\uparrow\rangle \quad e_8 = |\downarrow\downarrow\downarrow\rangle.$$

This means that e_1, e_4, e_6 and e_7 are even, whereas e_2, e_3, e_5 and e_8 are odd. It follows that

$$\mathcal{R}(u, v)\mathcal{T}(u) \otimes_s \mathcal{T}(v) = \mathcal{T}(v) \otimes_s \mathcal{T}(u)\mathcal{R}(u, v) \tag{19}$$

where $\mathcal{T}(u)$ is the monodromy matrix

$$\mathcal{T}(u) = \mathcal{L}_{0L}(u) \cdots \mathcal{L}_{0L}(u). \tag{20}$$

The graded Yang–Baxter algebra (19) ensures the commutativity of the transfer matrix $\tau(u) = \mathrm{Str}\,\mathcal{T}(u)$ for different spectral parameters, i.e. $[\tau(u), \tau(v)] = 0$. This implies that $\tau(u)$ can be viewed as a generating function of an infinite number of commuting conserved currents, which may be obtained through the expansion of the $\tau(u)$ in powers of u

$$\ln\tau(u) = \ln\tau(0) + H\,u + \tfrac{1}{2}J\,u^2 + \cdots \tag{21}$$

where J is the first non-trivial conserved current next to the Hamiltonian

$$
\begin{aligned}
(-\mathrm{i})J = \sum_{j=1}^{L} \Bigg\{ & \sum_{\alpha=1}^{3} \left(c^\dagger_{j+1(\alpha)} c_{j-1(\alpha)} - c^\dagger_{j-1(\alpha)} c_{j+1(\alpha)} \right) \\
& \times \exp\left(\eta \sum_{\alpha' \neq \alpha} n_{j+\theta(\alpha'-\alpha)(\alpha')} \right) \exp\left(\eta \sum_{\alpha' \neq \alpha} n_{j-1+\theta(\alpha'-\alpha)(\alpha')} \right) \\
& - \exp\eta \, \sinh\eta \sum_{\alpha<\beta}^{3} \Big[\left(c^\dagger_{j-1(\alpha)} c_{j(\alpha)} - c^\dagger_{j(\alpha)} c_{j-1(\alpha)} \right) \left(c^\dagger_{j(\beta)} c_{j+1(\beta)} + c^\dagger_{j+1(\beta)} c_{j(\beta)} \right) \\
& + \left(c^\dagger_{j-1(\alpha)} c_{j(\alpha)} + c^\dagger_{j(\alpha)} c_{j-1(\alpha)} \right) \left(c^\dagger_{j(\beta)} c_{j+1(\beta)} - c^\dagger_{j+1(\beta)} c_{j(\beta)} \right) \Big] \\
& \times \exp\left(\eta \sum_{\alpha' \neq \alpha,\beta} n_{j-1+\theta(\alpha'-\alpha)(\alpha')} \right) \exp\left(\eta \sum_{\beta' \neq \alpha,\beta} n_{j+\theta(\beta'-\beta)(\beta')} \right) \\
& - \exp\eta \, \sinh\eta \sum_{\alpha<\beta}^{3} \Big[\left(c^\dagger_{j(\alpha)} c_{j+1(\alpha)} + c^\dagger_{j+1(\alpha)} c_{j(\alpha)} \right) \left(c^\dagger_{j(\beta)} c_{j+1(\beta)} - c^\dagger_{j+1(\beta)} c_{j(\beta)} \right) \\
& + \left(c^\dagger_{j(\alpha)} c_{j+1(\alpha)} - c^\dagger_{j(\alpha)} c_{j-1(\alpha)} \right) \left(c^\dagger_{j(\beta)} c_{j+1(\beta)} + c^\dagger_{j+1(\beta)} c_{j(\beta)} \right) \Big] \\
& \times \exp\left(\eta \sum_{\alpha' \neq \alpha,\beta} n_{j+\theta(\alpha'-\alpha)(\alpha')} \right) \exp\left(\eta \sum_{\beta' \neq \alpha,\beta} n_{j+\theta(\beta'-\beta)(\beta')} \right) \Bigg\}. \qquad (22)
\end{aligned}
$$

Therefore, we have built up an important ingredient towards the QISM approach for the model. Next, we shall discuss the integrable boundary conditions for the fermion model with the Hamiltonian density (11). The boundary conditions are known to be useful for studying conductivity properties in such non-fermion liquids (see, e.g., [13, 14]). The open boundary conditions for the spin model (2) were studied in [15, 16]. Now we show that the fermion version of $\mathcal{R}$ satisfies the following graded reflection equations

$$\mathcal{R}_{12}(u,v) \overset{1}{K}_{-}(u) \mathcal{R}_{21}(v,-u) \overset{2}{K}_{-}(v) = \overset{2}{K}_{-}(v) \mathcal{R}_{12}(u,-v) \overset{1}{K}_{-}(u) \mathcal{R}_{21}(-v,-u) \qquad (23)$$

$$\mathcal{R}_{21}^{\mathrm{St}_1\mathrm{St}_2}(v,u) \overset{1}{K}{}_{+}^{\mathrm{St}_1}(u) \tilde{\mathcal{R}}_{12}(-u,v) \overset{2}{K}{}_{+}^{\mathrm{St}_2}(v) = \overset{2}{K}{}_{+}^{\mathrm{St}_2}(v) \tilde{\mathcal{R}}_{21}(-v,u) \overset{1}{K}{}_{+}^{\mathrm{St}_1}(u) \mathcal{R}_{12}^{\mathrm{St}_1\mathrm{St}_2}(-u,-v) \qquad (24)$$

using the conventional notation

$$\overset{1}{X} \equiv X \otimes_s \mathbf{I}_{V_2} \qquad \overset{2}{X} \equiv \mathbf{I}_{V_1} \otimes_s X \qquad (25)$$

where $\mathbf{I}_V$ denotes the identity operator on V, and, as usual, $\mathcal{R}_{12} = \mathcal{P} \cdot \mathcal{R}$ and $\mathcal{R}_{21} = \mathcal{P} \cdot \mathcal{R}_{12} \cdot \mathcal{P}$. Here $\mathcal{P}$ is the graded permutation operator which can be represented by a 64×64 matrix, i.e.

$$P_{\alpha\beta,\gamma\delta} = (-1)^{P(\alpha)P(\beta)} \delta_{\alpha\delta} \delta_{\beta\gamma}. \qquad (26)$$

Furthermore, superscripts St_a and $\overline{\mathrm{St}}_a$ denote the supertransposition in the space with index a and its inverse, respectively

$$(A_{ij})^{\mathrm{St}} = (-1)^{[P(i)+P(j)]P(i)} A_{ji} \qquad (A_{ij})^{\overline{\mathrm{St}}} = (-1)^{[P(i)+P(j)]P(j)} A_{ji}. \qquad (27)$$

The graded reflection equations (23) and (24) together with the graded Yang–Baxter algebra (19) and the following properties

$$\mathcal{R}_{12}(u,v) \mathcal{R}_{21}(v,u) = 1 \qquad (28)$$

$$\tilde{\mathcal{R}}_{21}^{\mathrm{St}_1}(-v,u)\mathcal{R}_{12}^{\mathrm{St}_2}(u,-v)=1 \tag{29}$$

$$\tilde{\mathcal{R}}_{12}^{\mathrm{St}_2}(-u,v)\mathcal{R}_{21}^{\mathrm{St}_1}(v,-u)=1 \tag{30}$$

assure that the double-row transfer matrix

$$\tau(u)=\mathrm{Str}_0\,K_+(u)\mathcal{T}(u)K_-(u)\mathcal{T}^{-1}(-u) \tag{31}$$

commutes for different spectral parameters, proving the integrability of the model with open boundary conditions. After a lengthy calculation, we find that the left boundary $K_-(u)$-matrix is given by

$$K_-^{(m)}(u)=\frac{1}{\lambda_-}\begin{pmatrix} A_-(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_-(u) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_-(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_-(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_-(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_-(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_-(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_-(u) \end{pmatrix} \tag{32}$$

where for $m=1$ we have

$$\begin{aligned}
A_-(u)&=(c_-+u)\left(\mathrm{e}^{2\eta}c_-+u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
B_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_-+u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
C_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_-+u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
D_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_--u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
E_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_-+u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
F_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_--u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
G_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_--u\right)\left(\mathrm{e}^{4\eta}c_-+u\right)\\
H_-(u)&=(c_--u)\left(\mathrm{e}^{2\eta}c_--u\right)\left(\mathrm{e}^{4\eta}c_--u\right)\\
\lambda_-&=\frac{1}{\mathrm{e}^{6\eta}c_-^3}
\end{aligned}$$

while, for $m=2$

$$\begin{aligned}
A_-(u)&=E_-(u)=(c_-+u)\left(c_-+\mathrm{e}^{2\eta}u\right)\\
B_-(u)&=C_-(u)=F_-(u)=G_-(u)=(c_-+u)\left(c_--\mathrm{e}^{2\eta}u\right)\\
D_-(u)&=H_-(u)=(c_--u)\left(c_--\mathrm{e}^{2\eta}\right)\\
\lambda_-&=\frac{1}{c_-^2}
\end{aligned}$$

and for $m=3$

$$\begin{aligned}
A_-(u)&=C_-(u)=E_-(u)=G_-(u)=(c_-+u)\\
B_-(u)&=D_-(u)=F_-(u)=H_-(u)=(c_--u)\\
\lambda_-&=\frac{1}{c_-}.
\end{aligned}$$

These results coincide with those obtained for the spin model studied in [16]. However, its companion, right boundary $K_+(u)$-matrix is given by

$$K_+^{(l)}(u) = \begin{pmatrix} A_+(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_+(u) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_+(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_+(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_+(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_+(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_+(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_+(u) \end{pmatrix} \tag{33}$$

where for $l = 1$, we have

$$\begin{aligned}
A_+(u) &= \left(e^{6\eta}c_+u - 1\right)\left(e^{4\eta}c_+u - 1\right)\left(e^{2\eta}c_+u - 1\right) \\
B_+(u) &= -e^{4\eta}\left(e^{2\eta}c_+u + 1\right)\left(e^{4\eta}c_+u - 1\right)\left(e^{2\eta}c_+u - 1\right) \\
C_+(u) &= -e^{2\eta}\left(e^{2\eta}c_+u + 1\right)\left(e^{4\eta}c_+u - 1\right)\left(e^{2\eta}c_+u - 1\right) \\
D_+(u) &= e^{6\eta}\left(e^{2\eta}c_+u + 1\right)(c_+u + 1)\left(e^{2\eta}c_+u - 1\right) \\
E_+(u) &= -\left(e^{2\eta}c_+u + 1\right)\left(e^{4\eta}c_+u - 1\right)\left(e^{2\eta}c_+u - 1\right) \\
F_+(u) &= e^{4\eta}(c_+u + 1)\left(e^{2\eta}c_+u + 1\right)\left(e^{2\eta}c_+u - 1\right) \\
G_+(u) &= e^{2\eta}(c_+u + 1)\left(e^{2\eta}c_+u + 1\right)\left(e^{2\eta}c_+u - 1\right) \\
H_+(u) &= -e^{4\eta}\left(c_+u + e^{2\eta}\right)(c_+u + 1)\left(e^{2\eta}c_+u + 1\right)
\end{aligned}$$

while for $l = 2$

$$\begin{aligned}
A_+(u) &= -B_+(u) = \left(e^{6\eta}c_+u - 1\right)\left(e^{4\eta}c_+u - 1\right) \\
C_+(u) &= -D_+(u) = -e^{2\eta}\left(e^{2\eta}c_+u + 1\right)\left(e^{4\eta}c_+u - 1\right) \\
E_+(u) &= -F_+(u) = -\left(e^{2\eta}c_+u + 1\right)\left(e^{4\eta}c_+u - 1\right) \\
G_+(u) &= -H_+(u) = e^{2\eta}(c_+u + 1)\left(e^{2\eta}c_+u + 1\right)
\end{aligned}$$

and for $l = 3$

$$\begin{aligned}
A_+(u) &= -B_+(u) = e^{2\eta}\left(e^{4\eta}c_+u - 1\right) \\
C_+(u) &= -D_+(u) = -\left(e^{4\eta}c_+u - 1\right) \\
E_+(u) &= -F_+(u) = -e^{2\eta}(c_+u + 1) \\
G_+(u) &= -H_+(u) = (c_+u + 1).
\end{aligned}$$

The above matrix is different from that in the non-graded case. Thus, the graded reflection equations (23) and (24) warrant the following boundary terms to be integrable

$$B_1^{(m)} = \begin{cases} \dfrac{1}{c_-\exp(2\eta)}\Bigg[\exp(-2\eta)\displaystyle\sum_{\alpha=1}^{3} n_{1(\alpha)} + 2\exp(-\eta)\sinh\eta \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3} n_{1(\alpha)}n_{1(\beta)} \\ \qquad + 4\sinh^2\eta\, n_{1(1)}n_{1(2)}n_{1(3)}\Bigg] & \text{for } m = 1 \\ \dfrac{\exp(\eta)}{c_-}\Bigg[\exp(-\eta)\displaystyle\sum_{\alpha=2}^{3} n_{1(\alpha)} + 2\sinh\eta\, n_{1(2)}n_{1(3)}\Bigg] & \text{for } m = 2 \\ \dfrac{1}{c_-}n_{1(3)} & \text{for } m = 3 \end{cases} \tag{34}$$

$$
B_L^{(l)} = \begin{cases} \dfrac{1}{c_+}\left[\exp(-2\eta)\displaystyle\sum_{\alpha=1}^{3} n_{L(\alpha)} + 2\exp(-\eta)\sinh\eta \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3} n_{L(\alpha)}n_{L(\beta)} \right. \\ \qquad \left. + \; 4\sinh^2\eta \; n_{L(1)}n_{1(2)}n_{L(3)} \right] & \text{for } l = 1 \\ \dfrac{1}{c_+\exp(\eta)}\left[\exp(-\eta)\displaystyle\sum_{\alpha=1}^{2} n_{L(\alpha)} + 2\sinh\eta n_{L(1)}n_{L(2)} \right] & \text{for } l = 2 \\ \dfrac{1}{c_+} n_{L(1)} & \text{for } l = 3 \end{cases} \tag{35}
$$

where $c_\pm$ are the parameters describing the boundary effects. With the different choices of the pair (m, l) $(m, l = 1, 2, 3)$, there exist nine classes of integrable boundary terms compatible with the integrability of model (11).

So far, we have performed the fermionization of the one-dimensional Bariev model of three coupled XY chains. By verifying the graded Yang–Baxter relation, the fermionic *L*-matrix and *R*-matrix are derived explicitly. Further, the integrable boundary conditions for the fermion model are discussed. We would like to mention that the fermionization scheme proposed in [4, 5] also appears possible to be adopted here through the use of the fusion procedure in the identification of the local Hamiltonians [17]. Our result provides a good starting point towards the algebraic Bethe ansatz for the model with both periodic and open boundary conditions by means of the graded QISM, which will probably be a hard task.

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